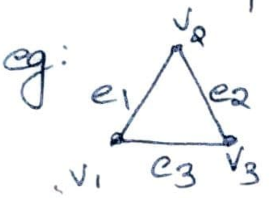
MODULE (5)

GRAPHS

Graphs & Graph models, Graph Terminology & special types of graphs, representing graphs and graph isomorphism, connectivity, Euler & Hamilton paths, shortest - path problems, planar graphs, graph coloring.

Def: A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \dots\}$ called vertices & another set $E = \{e_1, e_2, \dots\}$ whose elmts are called edges & each e_k is identified with unordered pair (v_i, v_j) of vertices.

eg:

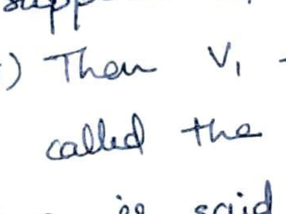


$V = \{v_1, v_2, v_3\}$

$E = \{e_1, e_2, e_3\}$

Directed graph or diagraph:

If each edge of a graph G has a direction then the graph G is called a directed graph



suppose $e_1 = (v_1, v_2)$ is an edge in a diagraph

(1) Then v_1 is called the initial vertex (origin or source) of e_1 & v_2 is called the terminal vertex of e_1 (or terminating or terminus)

(2) e_1 is said to be incident from v_1 & to be incident to v_2

(3) v_1 is adjacent to v_2 & v_2 is adjacent from v_1.

Undirected graph:

If each edge of the graph G has no direction then graph is called undirected graph.

Note: (1) No: of vertices in G is called order of a graph, $|V|$

(2) No: of edges in G is called size of a graph, $|E|$

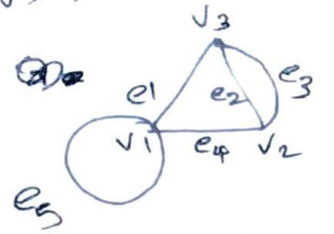
(3) A graph with an infinite vertex set is called infinite graph

(4) A graph with a finite vertex set is called a finite graph.

(5) An edge is called a self loop if terminal vertices are same.

(6) Parallel edges or multiple edges: In a graph if there are two or more edges associated with a given vertex (v_i, v_j), then the edge are referred to as parallel edges.

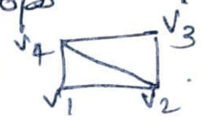
eg:



e_2 & e_3 are parallel edges

e_5 is a self loop.

(7) Simple graph: A graph without self loops and parallel edges is called a simple graph. eg:



(8) Multigraphs: Graphs that may have multiple edges connecting the same vertices are called multigraphs.

(9) Pseudographs: Graphs that may include loops and multiple edges connecting the same pair of vertices.

(10) Simple directed graph: When a directed graph has no loops & has no multiple directed edges it is called a simple directed graph.

(11) Directed multigraphs: Directed graphs that may have multiple directed edges from a vertex to a second vertex is called directed multigraphs.

(12) Multiplicity, m: when there are m directed edges, each associated to an ordered pair of vertices (u, v), then (u, v) is an edge of multiplicity m.

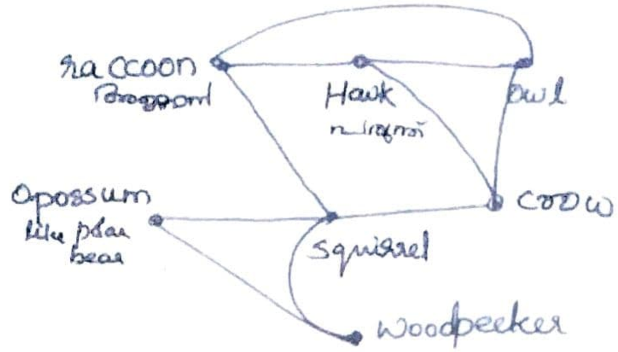
(13) Mixed graphs: A graph with both directed and undirected edges is called a mixed graph. which is used to model a computer network containings links that operate in both directions & other links that operate only in one direction.

Graph Models:

(1) Niche Overlap graphs in Ecology:

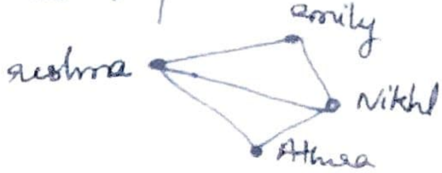
Graphs are used in many models involving the interaction of different species of animals. Each species is represented by a vertex. A niche overlap graph is a simple graph because no loops or multiple edges are needed in this model.

eg: ecosystem of a forest



(2) Acquaintanceship Graphs:

A graph models to represent various relationships between people. A simple graph can be used to represent whether two people know each other. eg:



(3) Influence graphs:

A directed graph called an influence graph can be used to model the behaviors of people. Each person is represented by a vertex. This graph does not contain loops and it doesnot contain multiple directed ~~graphs~~ edges.

HW (4) The Hollywood graph

(5) Round - Robin Tournaments

(6) Call graphs

(7) Collaboration graphs

Graph Terminology:

Def: (1) Two vertices u & v in an undirected graph G are called adjacent (or neighbors) in G if u & v are endpts of an edge of G.

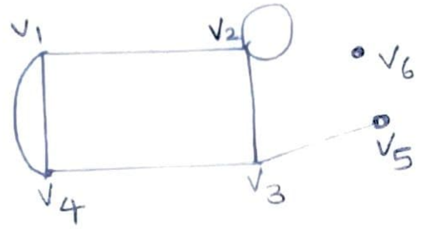
~~Def~~

Def: (2) The degree of a vertex in an undirected graph is the no: of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex and is denoted by deg(v)

Def: (3) A vertex of degree zero is called <u>isolated</u>.

A vertex is <u>pendant</u> $\Leftrightarrow$ it has degree one.

eg:



$d(v_1) = 3$, $d(v_2) = 4$

$d(v_3) = d(v_4) = 3$

$d(v_5) = 1$ $\quad d(v_6) = 0$

v_5 is pendant & v_6 is isolated.

Thm: (1) The Handshaking Thm:

Let $G = (V, E)$ be an undirected graph with e edges. Then

$$\sum_{v_i \in V} d(v_i) = 2e$$

Proof: Let us consider a graph $G = (V, E)$ with n vertices $v_1, v_2, \cdots, v_n$ and e edges. Then we have to ST

$$d(v_1) + d(v_2) + \cdots + d(v_n) = 2e.$$

Since each edge is associated with two vertices, it contributes two degree in graph G. Also, there are e edges in the graph & then the total degree of G will be equal to 2e.

Hence the sum of degrees of all vertices in G = 2e

$$\sum_{v_i \in V} d(v_i) = 2e$$

Q:(1) How many edges are there in a graph with 10 vertices each of degree 6 ?

By HST, $\sum d(V_i) = 2e$

$6 \times 10 = 2e \Rightarrow 60 = 2e \Rightarrow \underline{e = 30}$

Thr:(2) An undirected graph has an even no: of vertices of odd degree.

Proof: Let us consider a graph $G = (V, E)$ with n vertices and e edges.

Out of these n vertices, let us separate the vertices of even degree and vertices of odd degree.

So, the total degree of graph G can be calculated by taking the sum of degree of all even vertices & degree of all odd vertices

ie degree of $G = \sum_{\text{even}} \deg(V_i) + \sum_{\text{odd}} \deg(V_j)$

ie $2e = \sum_{\text{even}} \deg(V_i) + \sum_{\text{odd}} \deg V_j \longrightarrow (1)$

Since the LHS of (1) is even & the first part of RHS is also even, the second part of the RHS $\sum_{\text{odd}} \deg(V_j)$ must also be an even no:. Because all the terms in this sum are odd, there must be an even no: of such terms.

$\therefore$ there are an even no: of vertices of odd degree

Q:(2) A graph has 21 edges, 3 vertices of degree 4 & other vertices of degree 3. Find the no: of vertices in G.

Let the no: of vertices $= n$.

3 vertices are of degree 4 & remaining $(n-3)$ vertices are of degree 3.

Degree of $G = \sum_{\text{even}} d(V_i) + \sum_{\text{odd}} d(V_j) = 3 \times 4 + (n-3)3$

by HST, $2e = 12 + 3n - 9$

$\Rightarrow 2 \times 21 = 12 + 3n - 9$

$\Rightarrow 42 = 3 + 3n \Rightarrow 3n = 42 - 3 = 39 \Rightarrow \underline{n = 13}$

Q:(3) A graph G has 8 edges. Find the no: of vertices if the degree of each vertex is 2.

Let n = no: of vertices in G.

The degree of each vertex is 2 $\Rightarrow$ degree of $G = 2n$

$\Rightarrow 2e = 2n$

$\Rightarrow e = n$

$\Rightarrow \underline{n = 8}$ since e

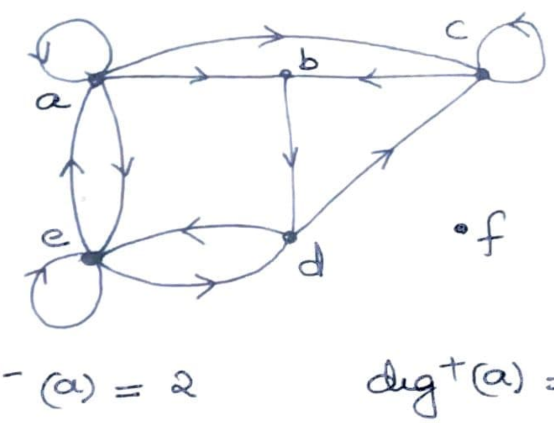
Def: In degree & out degree of a vertex

In a directed graph, the in-degree of a vertex v, denoted by $\deg^-(v)$ is the no: of edges with v as their terminal vertex.

The out-degree of v, denoted by $\deg^+(v)$ is the no: of edges with v as their initial vertex.

Note: a loop at a vertex contributes 1 to both the in-degree & out-degree of this vertex.

Q:(4) Find the in-degree & out-degree of each vertex in G.



$\deg^-(a) = 2$ $\quad$ $\deg^+(a) = 4$

$\deg^-(b) = 2$ $\quad$ $\deg^+(b) = 1$

$\deg^-(c) = 3$ $\quad$ $\deg^+(c) = 2$

$\deg^-(d) = 2$ $\quad$ $\deg^+(d) = 2$

$\deg^-(e) = 3$ $\quad$ $\deg^+(e) = 3$

$\deg^-(f) = 0$ $\quad$ $\deg^+(f) = 0$

Note: A vertex with zero indegree is called a source.
A vertex with zero outdegree is called a sink.

Some special simple graphs:

(1) Complete graphs

The complete graphs on n vertices denoted by K_n, is the simple graph that contains exactly one edge b/w each pair of distinct vertices.




etc

K_1 K_2 K_3 K_4 K_5 K_6

(2) Cycles

The cycles C_n, $n \geq 3$, consists of n vertices $1, 2, \ldots, n$ and edge $\{1,2\}$, $\{2,3\}, \ldots \{n-1, n\}$ & $\{n, 1\}$.



etc.

C_3 C_4 C_5

(3) Wheels.

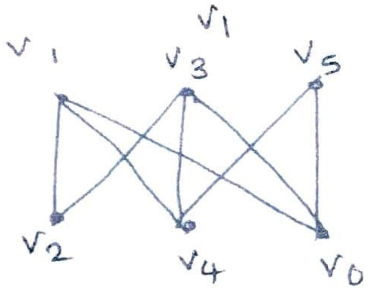
The wheel W_n is obtained by adding a vertex to the cycle C_n for $n \geq 3$.


etc.

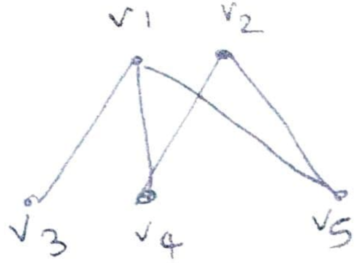
W_3 W_4

(4) Bipartite graphs:

A simple graph G is called bipartite if its vertex set V can be divided into two disjoint sets V_1 & V_2 s every edge in the graph connects a vertex in V_1 & a vertex in V_2 so that no edge in G connects either two vertices in V_1 or two vertices in

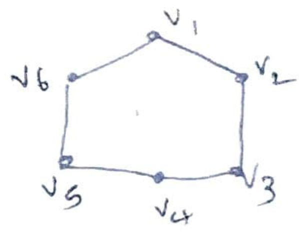


V_2



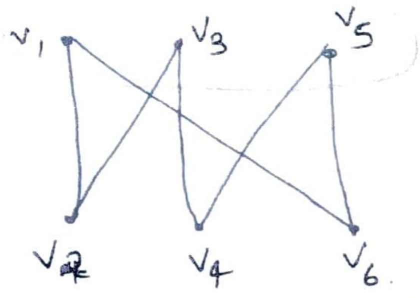
G1.

C_6 is bipartite ie



V_1

ie



V_2.

(5) Complete Bipartite graphs: $K_{m,n}$

is the graph that has its vertex set partitioned into two subsets m & n of vertices. There is an edge b/w two vertices $\Longleftrightarrow$ one vertex is in the first subset and other ~~two~~ vertex is in the second subset.

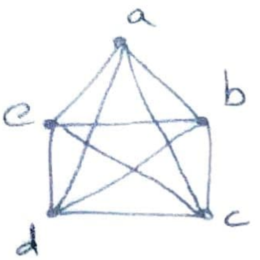
$K_{2,3}$

$K_{3,3}$

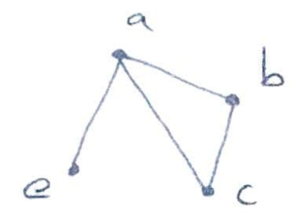
$K_{3,5}$

Subgraph:

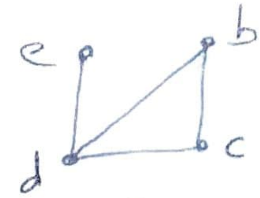
A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ & $F \subseteq E$. A subgraph H of G is a proper subgraph of G if $H \neq G$.



$G = K_5$



H_1



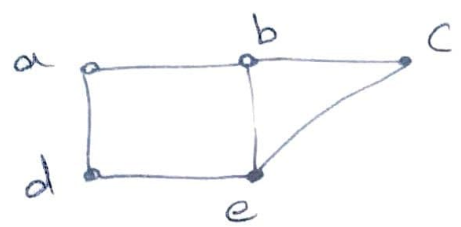
H_2

di

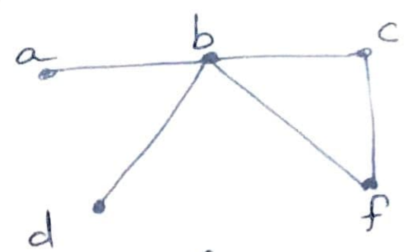
Union:

The union of two simple graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ & edge set $E_1 \cup E_2$. The union of G_1 & G_2 is denoted by $G_1 \cup G_2$.

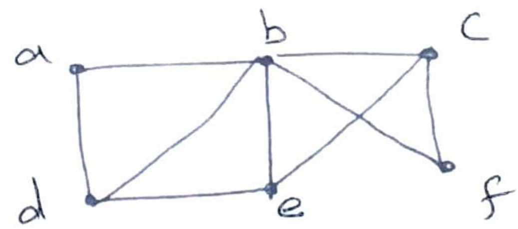
Q: Find the union of the graph G_1 & G_2



G_1



G_2



$G_1 \cup G_2$.

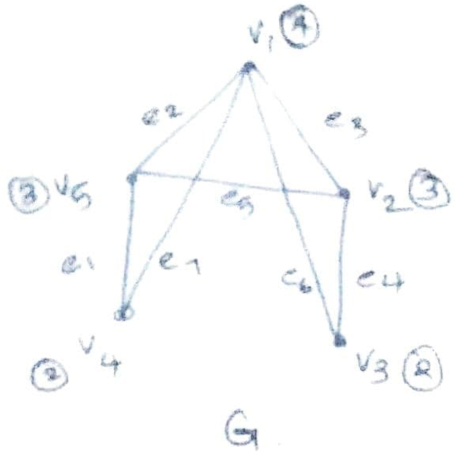
Q: Draw the graphs K_7, $K_{1,8}$, C_7, W_7, $K_{4,4}$

Degree Sequence: of a graph is the sequence of the degrees of the vertices of the graph in non increasing order.

Q How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2 ? Draw such a graph.

By HST $\sum d(v_i) = 2e$

$4+3+3+2+2 = 2e \implies 14 = 2e \implies e = \underline{\underline{7}}$



Q How many edges does a graph have if its degree seq is 5, 2, 2, 2, 2, 1 ? Draw such a graph.

Q: Draw all subgraph of

Regular graphs:

A simple graph is called regular if every vertex of this graph has the same degree.
A regular graph is called n-regular if every vertex in this graph has degree n

Q: For which values of n are these graphs regular ?

K_n – for every n – $(n-1)$ regular.

C_n – 2 - regular

W_n – 3-regular

Q: How many vertices does a regular graph of degree 4 with 10 edges have?

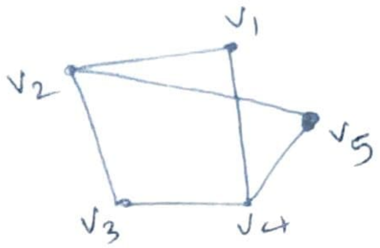
Let no. of vertices $= n$ $\qquad e = 10$

by HST $\quad \sum d(v_i) = 2e \implies 4n = 2 \times 10 \implies n = \frac{2 \times 10}{4} = \underline{\underline{5}}$

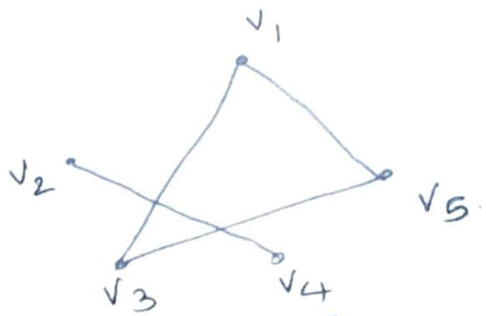
<u>Complementary graph</u> of loop-free undirected graph $G = (V, E)$ is $\overline{G} = (V, \overline{E})$ where V contains all the vertices of G & $\overline{E}$ is the set of edges defined as

$\overline{E} = \{ (v_i, v_j) \mid (v_i, v_j) \notin E \}$.

ie if 2 vertices are not adj in G, they will be adj in $\overline{G}$.



G



$\overline{G}$

$G = K_3$

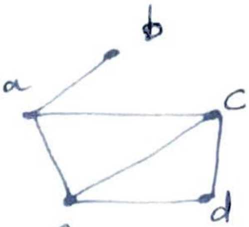
$\overline{G} = \overline{K_3}$ is a null graph (which has no edges)

<u>Representing graphs:</u>

(1) Adjacency lists:

One way to represent a graph without multiple edges is to list all the edges of this graph.

g:



Adjacency list:

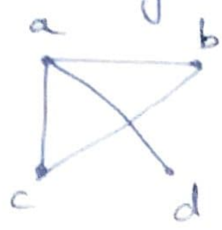
Vertex	Adj vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

(2) Adjacency matrix:

$G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $1, 2, \dots, n$. The adjacency matrix A or A_G of G wrt this listing of the vertices is the $n \times n$ zero-one matrix with 1 as its $(i,j)^{th}$ entry when i & j are adjacent & 0 as its $(i,j)^{th}$ entry when they are not adjacent.

i.e. adjacency matrix $A = [a_{ij}]$ then $a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$

eg: ①

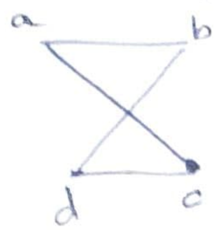


G_1.

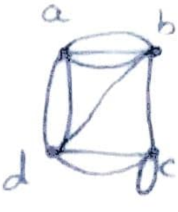
$$A_{G_1} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 1 & 0 & 0 \\ d & 1 & 0 & 0 & 0 \end{array}$$

② Draw a graph with adjacency matrix $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ wrt the ordering of vertices a, b, c, d.

ans:



③ Use an adjacency matrix to rep the pseudograph

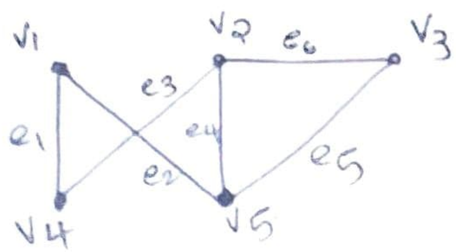


$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

(3) Incidence Matrices:

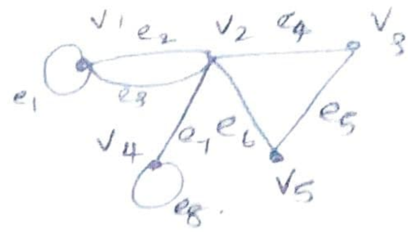
$G = (V, E)$ be an undirected graph. Suppose that $1, 2, \dots, n$ are the vertices & $e_1, e_2, \dots, e_m$ are the edges of G. Then the incidence matrix wrt ~~to this~~ V & E is the $n \times m$ matrix $M = [m_{ij}]$ where $m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } i \\ 0 & \rightarrow \text{otherwise} \end{cases}$

eg:



$$M = \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 1 & 1 & 0 & 1 \\ v_3 & 0 & 0 & 0 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_5 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}$$

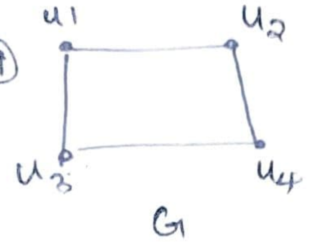
② Represent the incidence matrix of



Graph Isomorphism:

The simple graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are isomorphic if there is a one-one & onto function f from V_1 to V_2 with the property that a & b are adjacent in $G_1 \iff f(a)$ & $f(b)$ are adjacent in G_2, for all a & b in V_1. Such a fn f is called an isomorphism.

eg. ①



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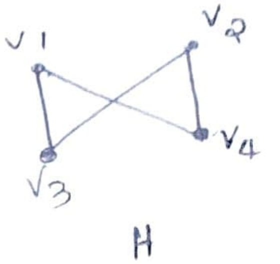


H

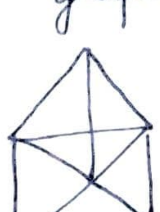
$f(u_1) = v_1$
$f(u_2) = v_4$
$f(u_3) = v_3$
$f(u_4) = v_2$

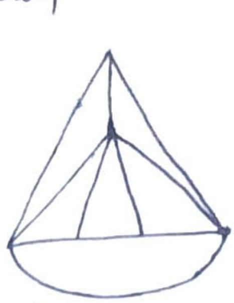
1-1 ~~one-one~~ & onto.

In G, u_1 & u_2, u_1 & u_3, u_2 & u_4, u_3 & u_4 are adjacent.

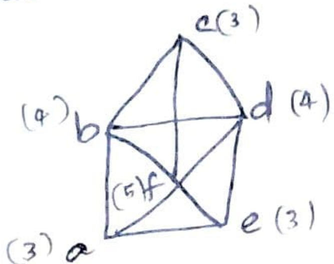
and $f(u_1) = v_1$ & $f(u_2) = v_4$
$f(u_1) = v_1$ & $f(u_3) = v_3$
$f(u_2) = v_4$ & $f(u_4) = v_2$
$f(u_3) = v_3$ & $f(u_4) = v_2$ are adj in H.

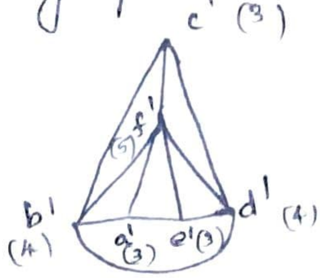
② ST graphs are isomorphic to each other.





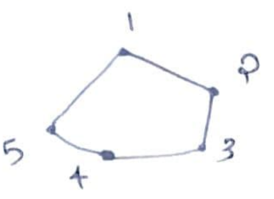
ans. Let us label the graphs



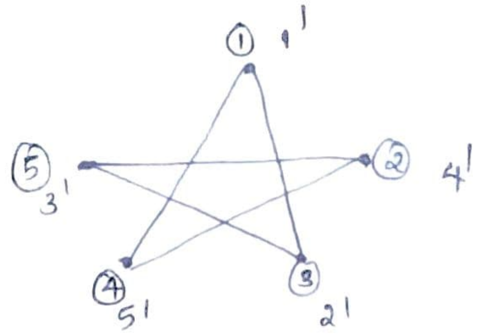


The no: of vertices & edges are equal in both the graphs & both graphs are having 3 vertices of degree 3, 2 of degree 4 & 1 vertex of deg 5.
There is a 1-1 & onto correspondance b/w the edges of two graphs preserving incidence relationship.
$(a,b) \leftrightarrow (a',b')$, $(a,e) \leftrightarrow (a',e')$, $(a,f) \leftrightarrow (a',f')$ & so on
so the ~~gr~~ graphs are isomorphic.

(3) Draw a cycle graph which is isomorphic to its complement.



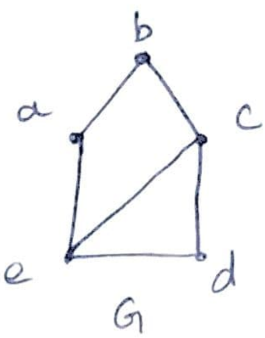
C_5



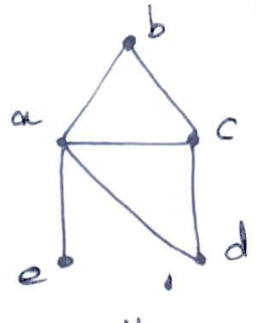
Here 5' & 2' are adjacent to 1' in G' , while 5 & 2 are adjacent to 1 in G. 3' & 1' are adj to 2' in G' while 3 & 1 are adj to 2 in G.

Also $\deg(i') = \deg(i)$ $\forall i$
Hence G & G' are isomorphic.

(4) ST G & H are not isomorphic.



G



H

Both G & H have 5 vertices & 6 edges. But H has a vertex of degree 1 (for e), whereas G has no vertices of degree 1.
$\Rightarrow$ G & H are not isomorphic.

Connectivity:

Walk is a finite alternating sequence of vertices & edges $v_1, e_1, v_2, e_2, \ldots, e_m, v_k$ starting & ending with vertices & each edge in the sequence is incident with the vertices following & preceding it.
No edges appears more than once in a walk.

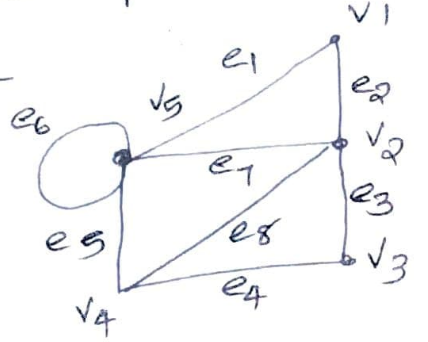
Open & closed walk: If a walk begins & ends at the same vertex the walk is called a closed walk. Otherwise open walk (ie terminal vertices are diff.)

Trial: is a sequence of vertices where no edge is repeated

Path: is a sequence of vertices where no vertex is repeated or an open walk in which no vertex appears more than once is called a path.
The total no. of edges appearing in a path is called the length of path. An edge which is not a self loop in a graph is a path of length one. The degree of the terminal vertices in a path is one & all the intermediate vertices will have degree two.

Note: loops can be included in a walk but not in a path.

eg:



walk : $v_1 e_2 v_2 e_3 v_3 e_4 v_4 e_5 v_5$
$v_5 v_2 v_4 v_3$
$v_1 v_5 v_5$ not a path, because of self loop

open walk : $v_5 v_1 v_2 v_3 v_4$

closed walk : $v_1 v_2 v_3 v_4 v_5 v_1$

Path : $v_1 v_5 v_2 v_4 v_3$

Trial : $v_1 v_2 v_3 v_4 v_2 v_5$

Circuit: (Cycle, elementary cycle, circular path or polygon)
A closed walk in which no vertex (except the initial & ~~final~~ final vertex) appears more than once is called a circuit. eg: $v_5 e_6 v_5$, $v_2 v_3 v_4 v_2$.

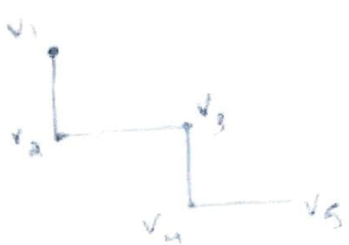
Note (1) Every self loop is a circuit but not vice-versa

(2) The min length of a path from v_i to v_j is called geodesic

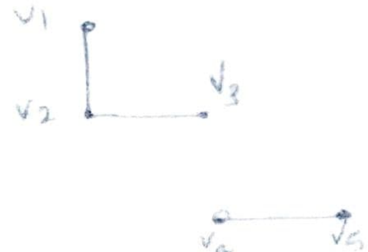
(3) A path or circuit is simple if it does not contain the same edge more than once

Connected graphs, Disconnected graphs & Components.

~~A graph~~ An undirected graph is called connected, if there is a path from any vertex v_i to v_j or vice versa. Otherwise the graph is disconnected



Connected



Disconnected

The disconnected graph consists of two connected graphs. Each of these is a connected ~~of~~ subgraph of the graph G. So a disconnected graph consists of 2 or more connected graphs & each of the connected subgraph is called component of the disconnected graph.

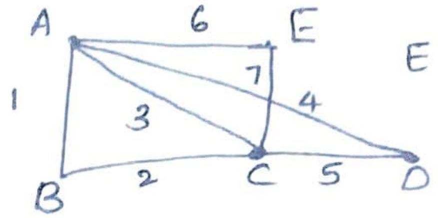
Note A null graph of n vertices is a disconnected graph consisting of n-components, each component containing one vertex & no edges.

Euler Circuit & Trail:

Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G is said to have an Euler circuit if there is a circuit in G that traverses every edge of the graph exactly once.

If there is an open trail from a to b in G and this trail traverses each edge in G exactly once the trail is called an Euler trail.

eg:



EC - A 1 B 2 C 3 A 6 E 7 C 5 D 4 A

G is an Euler graph if it contains an Euler circuit in G.

Thm: Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G has an Euler circuit $\Longleftrightarrow$ G is connected and every vertex in G has even degree.

OR

A graph G is an Euler graph iff all the vertices of G are of even degree.

Proof: Suppose that G is an Euler graph. Then it contains an Euler circuit.

In tracing this walk, we observe that everytime the walk meets a vertex v, it goes through two new edges incident on v.

For eg: while tracing a walk, when the walk meets a new vertex say v_2, it goes through two edges e_1 & e_2 which are incident on v_2 with e_1 we entered v_2 and with e_2 we exited. This is true with the terminal vertex also as we exited and entered the same vertex at the starting and end of the walk resp. So everytime we trace a vertex, its degree increase by 2. Thus if G is an Euler graph the degree of every vertex is even.

Conversely, let us assume that all the vertices of G are of even degree. TPT G is an Euler graph we need to construct an Euler circuit in G.

Now we construct a walk starting at an arbitrary vertex v and going through the edges of G s.t no edges is traced more than once.

Since G contains all the vertices of even degree, we can exit from vertex we entered (closed).

But the tracing cannot stop at any other vertex other than v since v is also of even degree.

∴ the walk we have constructed is a closed walk in which each edge is included exactly once.

∴ G has an Euler circuit.

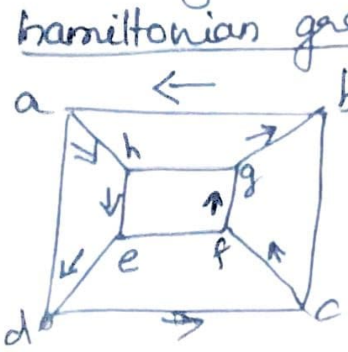
∴ G is an Euler graph.

Hamiltonian paths & cycles:

A hamiltonian circuit or cycle (HC) in a connected graph is defined as a closed walk that traverses every vertex of G exactly once, except of course the starting vertex, at which the walk also terminates.

A hamiltonian graph G is the graph which contains a HC

eg:

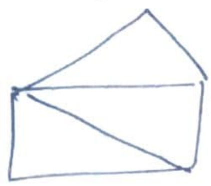


HC - a h e d c f g b a

If we remove any edge from a HC, we are left with a path called Hamiltonian path (H.P) clearly a HP in G traverses every vertex of G.

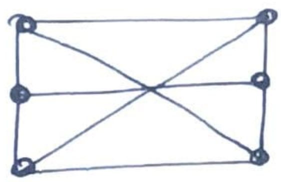
Planar Graphs:

A graph G is said to be planar if there exist some geometric representation of the graph G which can be drawn on a plane without crossover between its edges.



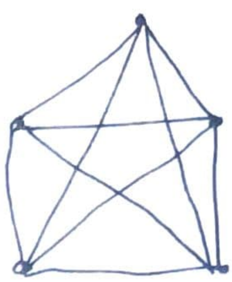
Non-planar graphs:

A graph that cannot be drawn on a plane without a crossover b/w the edges is called a non-planar graph.

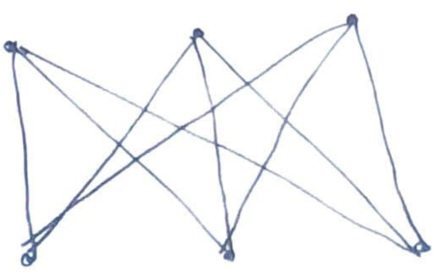


Kuratowski's graphs:

Kuratowski introduced the two non-planar graphs called Kuratowski's graphs. They are K_5 & $K_{3,3}$



K_5



$K_{3,3}$

Properties of Kuratowski's graphs:

(1) Any graph isomorphic to any of the Kuratowski's graph is non-planar.

(2) If a graph G contains any of the Kuratowski's graph then G is non-planar.

(3) Both K_5 & $K_{3,3}$ are regular graphs.

(4) K_5 is non-planar with the smallest no: of vertices.

(5) $K_{3,3}$ is non-planar with the smallest no: of edges.

(6) Removal of one vertex or one edge makes the graph planar.

~~Euler's Formula:~~

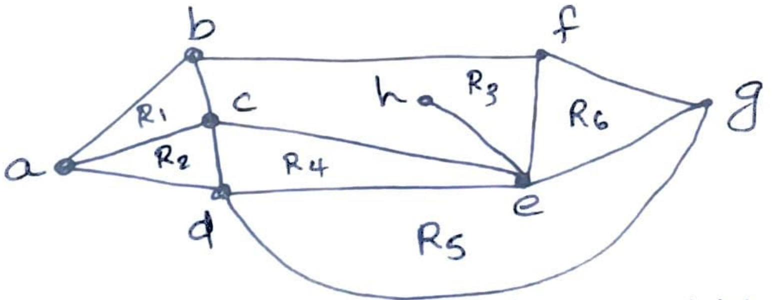
~~For any connected graph G with n vertices and e-edges the total no: of regions r is~~

Region and its degree:

The planar representation of graph divides the plane of paper into several regions, also called faces or meshes or windows.

The degree of any region say R denoted as deg (R) is the length of the closed walk which bounds the region R.

eg:



deg R_3 = 6, bfehecb
(e,h) is occuring twice.

deg R_4 = 3

deg R_5 = 3

deg R_6 = 3

deg R_7 = 5 dgfbad
outer region

deg (R_1) = length of closed walk which bounds R_1 = 3

deg R_2 = 3

Note: (1) Region is not defined for a nonplanar graph

(2) Sum of the degree of the regions is equal to twice the no: of edges. (same as that of Handshaking Thm)

Euler's Formula:

For any connected graph G with n vertices & e edges the total no: of regions $r = e - n + 2$ or $v - e + r = 2$

Proof:

We prove the formula by induction on n.

When $n = 1$, then G is a bouquet of loops.

clearly $r = e + 1$

eg: $e = 1$  $e = 2$ 

$r = 2$ $r = 3$

$\therefore$ The formula is true for $n = 1$

Assume the result is true for connected plane graphs with $(n-1)$ vertices. Then TPT it is true for n vertices.

Let G be a connected planar graph with n vertices, e edges & r regions, $n > 1$.

Since G is connected, $\exists$ atleast one edge in G which is not a loop.

Let x be such an edge. 

Let $G' = G - x$ (contracting edge) 

No: of edges in $G' = e - 1$

" regions " $= r$

" vertices " $= n - 1$

$\therefore$ by assumption of induction

$(n-1) - (e-1) + r = 2$

$n - e + r = 2$

Hence the proof.

By our hypothesis the formula is valid for all graphs having r regions i.e. $n-e+r=2$

$\therefore$ RHS = LHS in ① $\Rightarrow$ $n'-e'+(r+1) = n-e+r = 2$

Hence the theorem

Result (1) In a simple connected graph with r regions, n vertices and e edges $(e>2)$ the following inequalities must hold

(i) $e \geq \frac{3}{2} r$ (ii) $e \leq 3n-6$.

Proof.

(i) Since $e>2$, this implies that each region is bounded atleast 3 edges and each edge belongs to exactly two regions.

The total degree of all the regions $= 2e \rightarrow$ ①

Since each region is bounded by atleast 3 edges, degree of one region ≥ 3.

$\therefore$ Degree of r regions $\geq 3r$ = Total degree of all the regions $\rightarrow$ ②

by ① & ② $2e \geq 3r$

$$e \geq \frac{3r}{2}.$$

(ii) By Euler's formula, $r = 2-n+e$

So in-equality (i) becomes

$$e \geq \frac{3}{2}(2-n+e)$$

$$\Rightarrow 2e \geq 6-3n+3e$$

$$\Rightarrow 3n-6 \leq e$$

$$\Rightarrow e \leq 3n-6$$

Result (2)

If G is a connected simple planar graph with n vertices $(n \geq 3)$ and e edges and no circuits of length 3, then
no triangles
$e \leq 2n-4$.

Proof:

Let there be r regions, each region is bdd by atleast 4 edges. $\because n \geq 3$

$$4r \leq 2e \Rightarrow r \leq \frac{2e}{4}$$

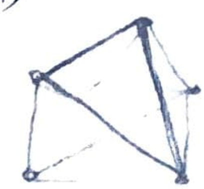
$n - e + r = 2$ by Euler's formula.

$$n - e + \frac{2e}{4} \geq 2$$

$$2n - e \geq 4$$

$$e \leq 2n - 4$$

(1) PT K_5 is non planar. eg: of Result (1)



K_5

$K_n = K_5 \Rightarrow n = 5$

$\therefore e = 10$

$\therefore$ ~~$3n - 6 = 3 \times 5 - 6 = 9$~~

K_5 is a simple connected ~~planar~~ graph and the smallest length for any cycle is 3. Suppose that the graph is planar
$\therefore$ By a previous result, $e \leq 3n-6$.

Here $3n-6 = 3 \times 5 - 6 = 9 \ngeq 10$ which is a contradiction

$\therefore K_5$ is non planar.

(2) ST $K_{3,3}$ is not planar. eg: of Result (2)

In a bipartite graph, there does not exist a circuit of length 3 and so $K_{3,3}$ has no circuit of length 3.

By previous result, $e \leq 2n-4$

Here $n = 6$, $e = 9$

$\therefore 2n - 4 = 8 \geq 9$ which is a $\Rightarrow\Leftarrow$ $\therefore K_{33}$ is non planar

(3) A connected planar graph has 10 vertices each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Here $n = 10$

$d(v_i) = 3$ for $i = 1, 2, \cdots, 10$

$$\therefore \sum_{i=1}^{10} d(v_i) = \sum_{i=1}^{10} 3 = 3 \times \sum_{i=1}^{10} 1 = 3 \times 10 = 30$$

$\therefore$ By handshaking thm, $\sum_{i=1}^{n} d(v_i) = 2e$

$\therefore 2e = 30 \Rightarrow e = 15$

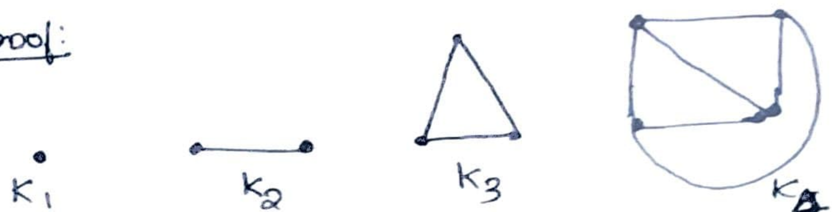
ie we have $n = 10$, $e = 15$

By Euler's formula, $n - e + r = 2$

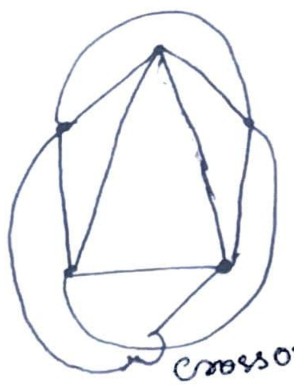
$r = 2 - n + e = 2 - 10 + 15 = \underline{7 \text{ regions}}$

(4) ST K_n is a planar graph for $n \leq 4$ and non planar graph for $n \geq 5$.

<u>Proof:</u>



The graphs K_1, K_2, K_3 & K_4 are by construction a planar graph since they do not contain any crossover between the edges.



K_5

It is not possible to draw this graph on a two dimensional plane without crossover between the edges. whatever way we draw, atleast one of the edges will cross over. Hence K_5 is not a planar graph.

∴ For any $n \geq 5$, K_n must contain a subgraph isomorphic to K_5. Since K_5 is not planar, any graph containing K_5 as its one of the subgraph cannot be planar.

(5) PT K_4 & $K_{2,2}$ are planar graphs.

In K_4, we have $n=4$ & $e=6$

We know that for a simple connected planar graph with n vertices $(n \geq 3)$ and e edges, then $e \leq 3n-6$

$3n-6 = 3 \times 4 - 6 = 6$

∴ $6 \leq 6$

∴ K_4 satisfies the inequality $e \leq 3n-6$.

∴ K_4 is a planar graph.

In $K_{2,2}$, we have $n=4$ & $e=4$

Also for a simple connected graph with n vertices and e edges and no circuit of length three, then $e \leq 2n-4$

$2n-4 = 2 \times 4 - 4 = 4$

∴ $e \leq 2n-4$

∴ $K_{2,2}$ satisfies the inequality $e \leq 2n-4$

∴ $K_{2,2}$ is a planar graph.

(6) If every region of a simple planar graph with n vertices and e edges embedded in a plane is bounded by k edges then S.T $e = \frac{k(n-2)}{k-2}$.

Let the degree of each region embedded in a plane be k

ie $d(R_i) = k$

∴ $\sum d(R_i) = \sum r = kr$ [Since the region is bdd by k edges]

Also $\sum d(R_i) = 2e$

∴ $kr = 2e$

∴ $r = \frac{2e}{k}$.

By Euler's formula, $n-e+f=2$

ie $n-e+\frac{2e}{k}=2$

$k(n-e)+2e=2k$

$nk-ke+2e=2k$

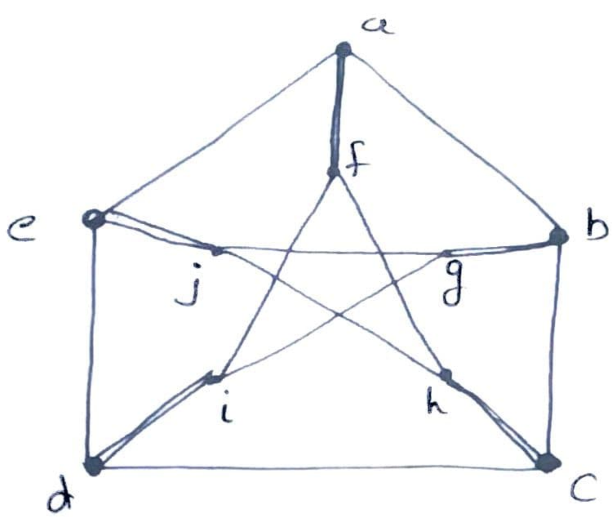
$nk-e(k-2)=2k$

$(nk-2k)=e(k-2)$

$k(n-2)=e(k-2)$

$$e=\frac{k(n-2)}{k-2}$$

(7) ST Peterson graph is not planar.



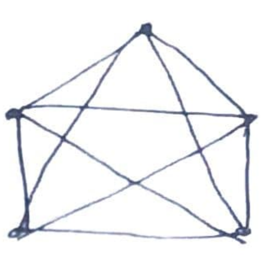
$n=10$

$e=15$

$d(n_i)=3$

[Thm: A finite graph G is planar $\Leftrightarrow$ it contains no subgraph that is edge-contractible to K_5 or $K_{3,3}$]

A peterson graph is one with 10 vertices and 15 edges and all the vertices is of degree 3. Contract the edges given as double lines to get a graph as below



K_5

Since the Peterson graph is edge contractible to K_5 it is not a planar graph.

(8) A connected graph has 9 vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4 & 5

(i) How many edges are there?

(ii) How many faces (regions) are there?

We know that $\sum_{i=1}^{n} d(v_i) = 2e$.

ie $2+2+2+3+3+3+4+4+5 = 2e$

$28 = 2e$

$\therefore e = \underline{\underline{14}}$

Thus $n = 9$, $e = 14$, $r = ?$

By Euler's formula, $r = e - n + 2$

$= 14 - 9 + 2$

$= \underline{\underline{7}}$

(9) Find a graph with degree sequence 4, 4, 3, 3, 3, 3 & G is planar

$\sum d(v_i) = 2e$

$4+4+3+3+3+3 = 2e \Rightarrow 20 = 2e \Rightarrow e = \underline{\underline{10}}$

For a simple connected planar graph with n vertices ($n \geq 3$) and e edges, then $e \leq 3n - 6$

$3n - 6 = 3 \times 6 - 6 = 12$

ie $e = 10 \leq 12 = 3n - 6$ ie $e \leq 3n - 6$

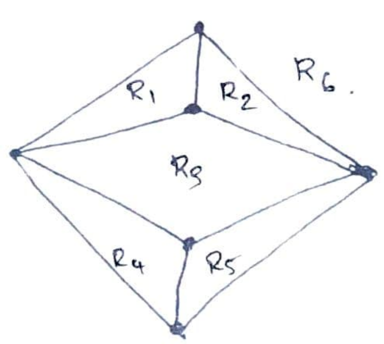
Thus the graph G satisfies the inequality $e \leq 3n - 6$

$\therefore$ G is a planar graph.

By Euler's formula; $n - e + r = 2$

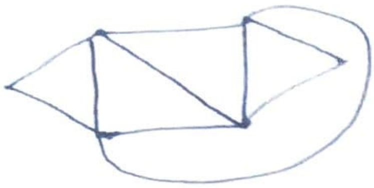
$\therefore r = e - n + 2$

$= 10 - 6 + 2 = 6$



(10) Determine the no. of regions defined by a connected ~~gr~~ planar graph, with 6 vertices and 10 edges. Draw a simple & non simple graph.

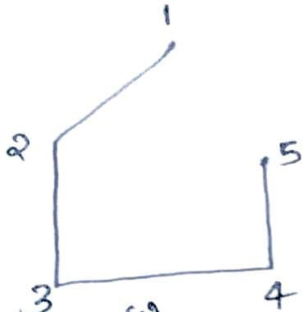
$e = 10$ & $n = 6$; $\therefore$ $r = e - n + 2 = 10 - 6 + 2 = \underline{\underline{6}}$



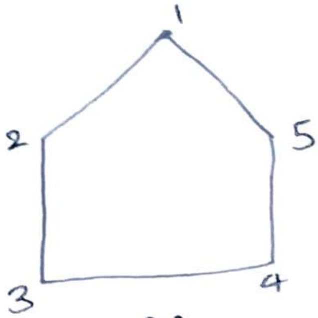
Simple graph

non - simple graph

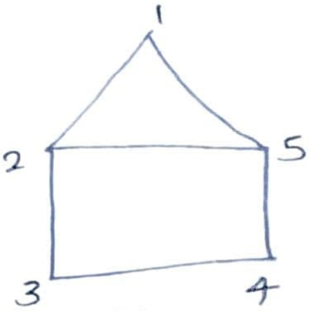
(11) Draw all planar graph with 5 vertices which are not isomorphic to each other.



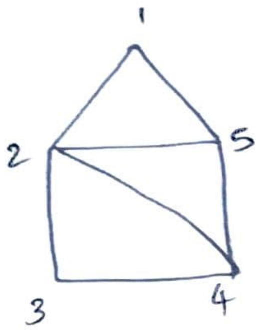
(1)



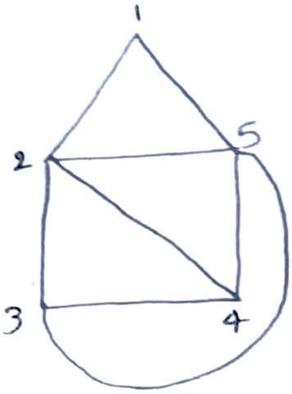
(2)



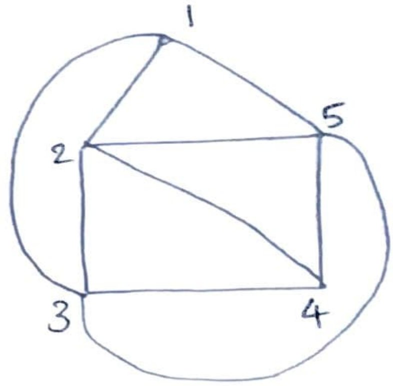
(3)



(4)



(5)

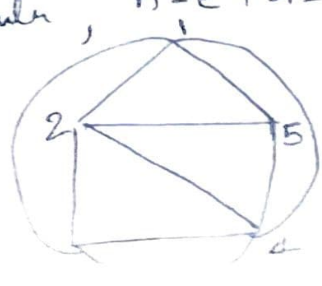


(6)

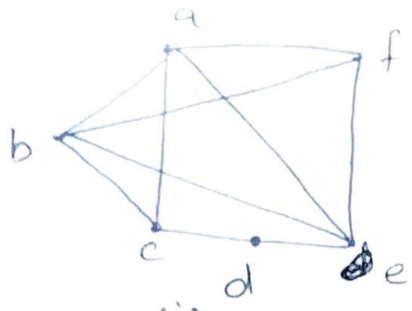
(12) How many edges must be a planar graph process if it has 7 regions & 5 vertices. Draw one such graph?

$r = 7$; $n = 5$

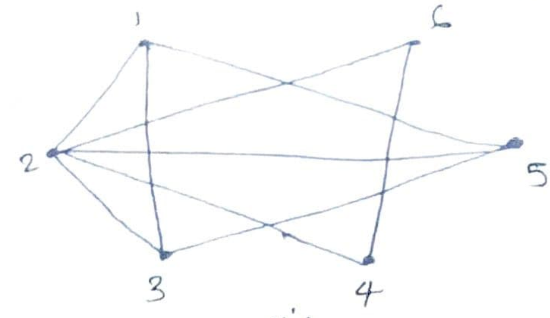
By Euler formula, $n - e + r = 2 \implies 5 - e + 7 = 2 \implies e = 10$



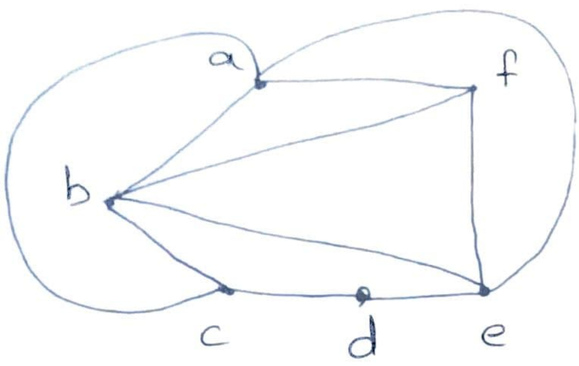
(13) By drawing the graph without any cross over bet edges, ST the following graphs are planar graph.



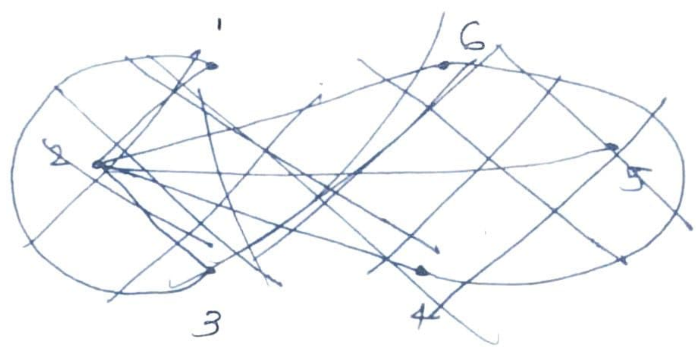
(i)

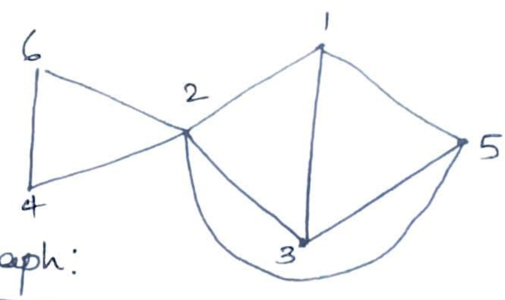


(ii)



(i)





Detection of Planarity of a graph:

Elementary reduction method:

Step 1: A disconnected graph is planar $\Longleftrightarrow$ each of its component is planar. We need to consider only one component at a time.

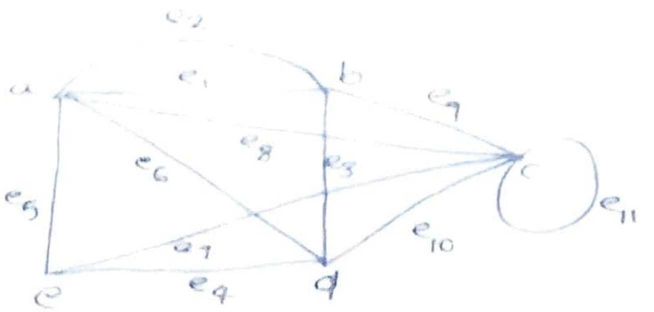
Step 2: Remove all self-loops as addition or removal of self loops does not affect planarity

Step 3: Remove all parallel edges as addition or removal of parallel edges doesnot affect planarity.

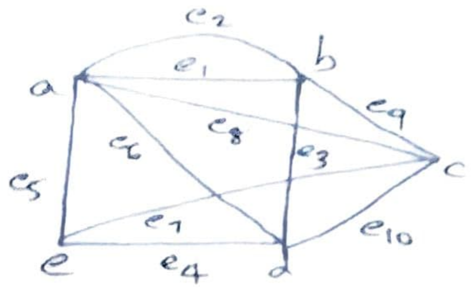
Step 4: Eliminate all the edges in series, as elimination of a vertex of degree two by merging two edges in series does not affect the planarity of the graph.

Step 5: Repeated application of step 3 & step 4 will eventually produce a graph.

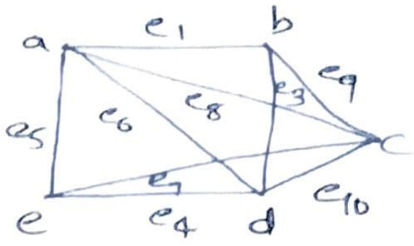
Check whether the graph is planar or not



By step 2: Removing all self loops from the gn graph, the new graph will be as follows



By step 3 Removing all parallel edges, the new graph is as follows



As the above graph does not contain any edge in series (ie vertex of degree 2), step 4 cannot be performed.

So the reduced graph by applying step 2 & step 3 is as above.

Now inorder to check the planarity of the above graph let's use Euler's ~~formula~~ inequality $e \leq 3n-6$

Here $n = 5$, $e = \cancel{9}\, 8$

$\therefore 3n-6 = 3\times5-6 = 15-6 = 9$

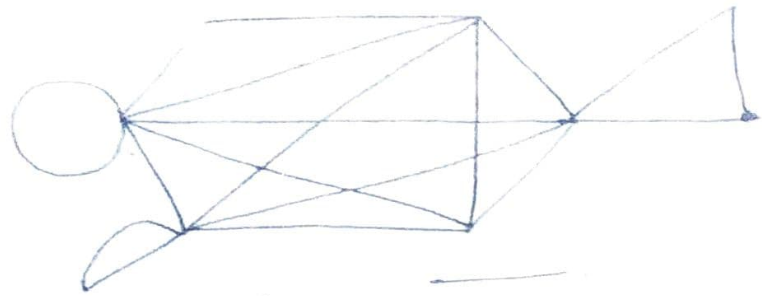
$\therefore e = 8 \leq 9 = 3n-6$

So the reduced graph satisfies the Euler's inequality for a planar graph

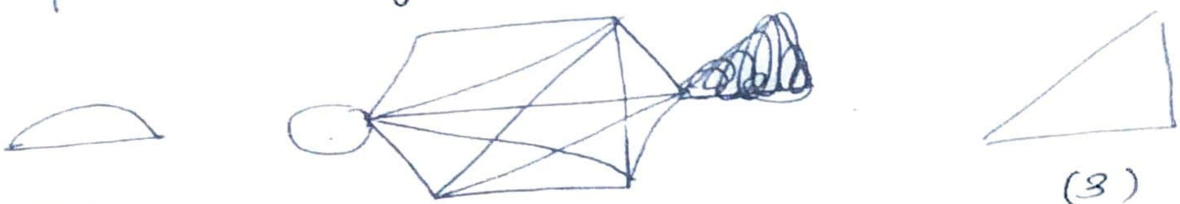
$\therefore$ The gn graph is planar.

Check the planarity of the given graph.

G :



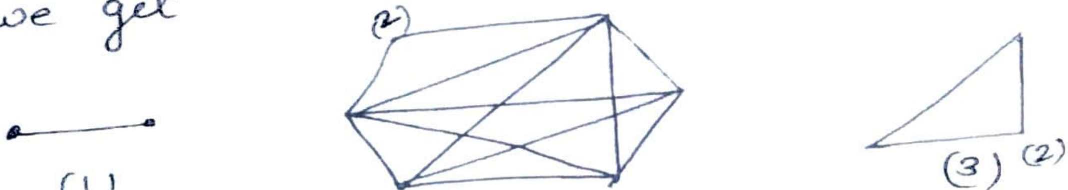
Separating blocks of G.



(1) (2) (3)

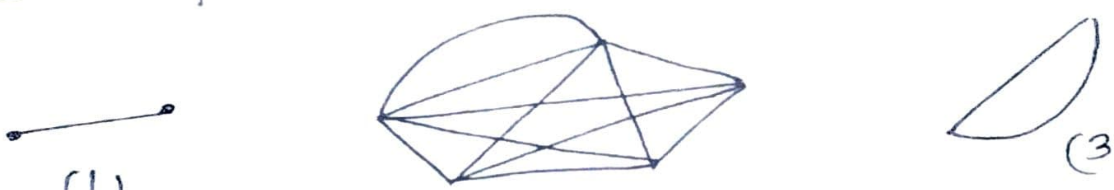
Eliminating parallel edges from block (1) & self loop from (2)

we get



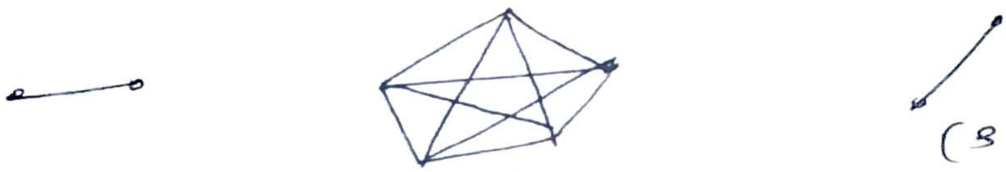
(1) (2) (3) (2)

Eliminating vertices of degree 2 from (2) & (3)



(1) (2) (3)

Eliminating parallel edges from (2) & (3)



(1) (2) (3)

The final reduced graph has 3 blocks of which, (1) & (3) a[re]
obviously planar. The second one (2) is the complete
graph K_5 which is non planar
as a subgraph and hence the given
∴ the given graph contains K_5
graph non-plan[ar]